On the universality of small scale turbulence

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1 Introduction

The complex behaviour of turbulence and especially small scale intermittency has been the subject of numerous investigations over the last 60 years and still the problem is not solved. Traditionally it is proposed that there exists a universal cascade process which is usually investigated by means of the so-called longitudinal velocity increment \( u(r) = e \cdot [v(x + er, t) - v(x, t)] \), and its transversal counterpart \( w(r) \). (\( v \) and \( e \) denote the velocity and an unit vector with arbitrary direction.) The statistics of \( u(r) \) and \( w(r) \) are often characterized by the structure functions \( S_n^u(r) = \langle u^n(r) \rangle \). Within the inertial range, the structure functions are commonly approximated by \( S_n^u(r) \propto r^{\zeta_n} \), respectively \( S_n^w(r) \propto r^{\xi_n} \).

In this work we present a more profound analysis which does not depend on a proposed scaling behavior. By pure data analysis we show evidence that the structure of small scale turbulence depends on the Reynolds number. Furthermore we show how the longitudinal and transversal velocity increments are coupled by a turbulent cascade process.

2 Stochastic Cascade Process

The object of our analysis is the general statistics of the velocity increments on different scales \( p(u_1, w_1, r_1; u_2, w_2, r_2; ...; u_N, w_N, r_N) \), where \( p \) denotes the joint probability density function and \( u_i \) denotes \( u(r_i) \). The idea is to characterize the structure of small scale turbulence by a stochastic cascade process which describes the evolution of the velocity increment with the scale \( r \). It has recently been shown \cite{1,2,3} that this evolution can be taken as a Markovian process with white noise, thus it can be described by a Fokker-Planck equation:

\[
-r \frac{\partial}{\partial r} p(u, r | u_0, r_0) = \left\{ -\frac{\partial}{\partial u} D^{(1)}(u, r) + \frac{\partial}{\partial u^2} D^{(2)}(u, r) \right\} p(u, r | u_0, r_0),
\]

(1)
here $r_0$ denotes a large scale and $r < r_0$. $D^{(1)}$ and $D^{(2)}$ are the drift and diffusion coefficient. The Markovian property implies that conditional probabilities $p(u, r | u_0, r_0)$ of (1) contain all information of the complete N-scale statistics. The coefficients $D^{(1)}$ and $D^{(2)}$ can be extracted directly from experimental data in a parameter free way by their mathematical definition (see [3]):

$$D^{(k)}(u, r) = \lim_{\Delta r \to 0} \frac{r}{k!\Delta r} \int_{-\infty}^{+\infty} (\tilde{u} - u)^k p(\tilde{u}, r - \Delta r | u, r) d\tilde{u}. \quad (2)$$

The corresponding formulation for higher dimensions, like for $u$ and $w$, can be found in [6],[7].

### 3 Nonuniversality for High Reynolds Numbers

We apply this method to analyze a set of measurements of the longitudinal velocity increments in a cryogenic free jet with a variation of the Reynolds number from 8500 to $10^6$ (max $R \lambda \approx 1200$). The drift and diffusion coefficients are given by: $D^{(1)}(u, r) = -\gamma(r)u$, $D^{(2)}(u, r) = \alpha(r) - \delta(r)u + \beta(r)u^2$. With the experimental estimation of the coefficients $D^{(1)}$ and $D^{(2)}$ the Fokker-Planck equation can be solved numerically and thus a quantitative verification of the quality of the estimated coefficients can be performed, see Fig. 1.

![Figure 1](image1.png)

**Fig. 1**: Contour plots of $p(u, r | u_0, r_0)$ for $r_0 = L$ and $r = 0.6L$. Comparison of the numerical solution (dashed lines) of the Fokker-Planck equation with the experimental data (solid lines), after [3].

![Figure 2](image2.png)

**Fig. 2**: Quadratic coefficient $\beta(r)$ of $D^{(2)}$ as functions of the scale $r / \lambda$ for Reynolds numbers increasing from bottom to top from 8500 to $10^6$. The solid curve corresponds to Kolmogorov’s four-fifth law.

The analysis of the Re - number dependence shows that in the limiting case $Re \to \infty$ these coefficients take the simple form:

$$D^{(1)}_{\infty}(u, r) = -\gamma(r)u, \quad D^{(2)}_{\infty}(u, r) = \beta_{\infty}(r)u^2. \quad (3)$$
For $\gamma(r)$ a universal function (indepent on Re) of $r/\lambda$ ($\lambda$ is the Taylor microscale) is found [4], [5], whereas $\beta(r)$ exhibits a clear dependence on the Reynolds number, see Fig 2.

With this result we can try to quantify how far away the measured results are from a supposed universal state. For this we consider the Kolmogorov’s four-fifth law of the third moment. After the multiplication of the Fokker–Planck equation (1) with $u^3$ from left and with $p(u_0,r_0)$ from right and successively integrating with respect to $u$ and $u_0$, the equation

$$r \frac{\partial}{\partial r} S^3_u(r) = 3(\gamma(r) - 2\beta_\infty(r))S^3_u(r) \quad (4)$$

is obtained.

According to Kolmogorov’s four-fifth law, $S^3_u(r) \propto r$, the left side of eq. (4) is equal to $1/3$. This condition for $\beta_\infty(r)$ is shown in Fig. 2. Thus nonuniversal changes in the statistics of small scale turbulence even for such high Reynold–numbers as $Re = 10^6$ are found.

### 4 Coupling between Longitudinal and Transversal Increments

Besides the one dimensional analysis of the longitudinal velocity increments we have also performed a corresponding two dimensional analysis of simultaneously measured longitudinal and transversal velocity increments. The measurements were performed by a X- hot wire anemometry in a turbulent wake behind a cylinder (Re = 12000 and $10^8$ data points were measured). In Fig. 3 the exemplary resulting drift and diffusion coefficients are shown. From our analysis we find that the drift coefficients do not couple the different increments: $D^{(1)}_{uw}(u,w,r) \propto f(u,r)$ and $D^{(1)}_{ww}(u,w,r) \propto g(w,r)$. The coupling of longitudinal and transversal increments is found in the diffusion coefficients only. $D^{(2)}_{uw}$ and $D^{(2)}_{ww}$ can be approximated roughly by $au^2 + bw^2$. As seen from Fig. 3c, $D^{(2)}_{uw}$ shows a clear coupling, it is close to a functional behavior of $D^{(2)}_{uw} \propto uw$.

### 5 Conclusion

Based on the mathematics of stochastic processes we were able to extract from measured data the functional form of stochastic processes describing the cascade process of small scale turbulence. The following results were obtained:

- The dependency of the stochastic process for the longitudinal velocity increments on the Reynolds number show clear non universality.
- The coupling between longitudinal and transversal velocity increments is due to the diffusion term of the stochastic process.
Universality of Turbulence

Figure 3: Drift and diffusion coefficients for longitudinal and transversal velocity increments estimated at $r = L/3$, where $L$ denotes the integral length.

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References