Fat tail statistics and beyond

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Abstract. Based on data from three different systems, namely, turbulence, financial market and surface roughness we discuss methods to analyze their complexities. Scaling analysis and fat tail statistics in the context of Lévy distributions are compared with a stochastic method, for which a Fokker-Planck equation can be estimated from the data. We show that the last method provides a more detailed characterization of the complexity.

1 Introduction

Complex systems whose variables are governed by nongaussian statistics, displaying fat tails, also called heavy tails, have gained considerable interest. On the one side, there is often the fundamental challenge to explain the mechanism leading to such a kind of statistics, as it is the case for turbulence, which is up to now one of the major unsolved probleme of physics. On the other side, such statistics have very important consequences for applications, as it is the case for risk management.

In this contribution we want to focus mainly on the problem of fat tails in the statistics of data from turbulence, from the financial market, and from the surface roughness. In each case the question is to characterize the disorder of the system by the statistics of a scale resolving quantity, $q(l,x)$, where l denotes the scale and x the location. Usually, properties of the quantity q are analyzed for different fixed scales l over the whole space x. Typical points of interest are to what kind of statistics the probability density (distribution) $p(q,l)$ belongs and what the functional l dependencies of general moments $< q(l,x)^n >$ are.

The question about the form of the distribution $p(q,l)$ is directly linked to the topic of fat tail distributions. There is the famous approach by Lévy statistics. Variables governed by Lévy statistics belong to stable processes, i.e. the random variables and the sum over these random variables have the same form of statistics. The crucial point of Lévy statistics $p_L(q)$ is that their fat tails do not decay exponentially fast as Gaussian distributions do. Lévy distributions decay with a power law for large or extreme events,

$$p_L(q) \cong \frac{1}{q^\beta} \qquad \text{for large} \quad q \quad \text{and} \quad 1 < \beta < 3. \tag{1}$$

It is easily seen that moments of the random variable $< q^n >$ diverge for $n > \beta - 1$. This is a quite unusual behavior. The reader may think over the implications of a system whose mean value is known but its variance is diverging. For example, what can we deduce from a results like the following: the probability of the magnitude of the next event is $< q > \pm\infty$. (For further reading we refer for example to [1].)

The other question of the functional l dependence of $< q(l,x)^n >$ is often connected with the question whether power law scalings is present. If $< q(l,x)^n > \propto l^{\xi_n}$ multiscaling or respectively multifractal properties are given. Per definition

$$< q^n >= \int_{-\infty}^{\infty} q^n p(q) dq, \tag{2}$$

the nth moment $< q^n >$ is directly linked to the probabilities $p(q)$. It is evident that if they obey Lévy laws, the question of multiscaling becomes inconsistent. For the case of turbulence we will show here, how these two aspects of multiscaling and fat tail distributions are connected.

So far we have reported on methods to analyze the disorder of a systems for separated scales. In this paper we want to report on an approach, which enables to achieve a more complete characterization. Namely, we aim to achieve the knowledge of the joint probabilities of finding different values of $q(l,x)$ for any scale $l : p(q, l_1; q, l_2; ....q, l_n)$. From this new approach we will see that the other approaches, like the multiscaling analysis, are not unique. For example, there are infinite many different ways to construct structures with the same multiscaling features.

2 Turbulence

The profound understanding of turbulence is up to now regarded as an unsolved problem. Although the basic equations of fluid dynamics, namely the Navier Stokes equations, are known for more than 150 years, a general solution of these equations for high Reynolds numbers, i.e. for turbulence, is not known. Even with the use of powerful computers no rigorous solutions can be obtained. Thus for a long time there has been the challenge to understand at least the complexity of an idealized turbulent situation, which is taken to be isotropic and homogeneous. This case will lead us to the well known intermittency problem of turbulence, which is nothing else than the occurrence of fat or heavy tailed statistics. The central question is to understand the mechanism which leads to this anomalous statistics (see [2,3]).

The intermittency problem of turbulence can be reduced to the question about the statistics of the velocity differences over different distances l, measured by the increments $q(l,x) = u(x+l) - u(x)$. Usually the velocity increments are taken from the velocity component in direction of the distance vector l. These are the so-called longitudinal velocity increments. By the use of energy considerations, a simple l-dependence of $q(l, x)$ was proposed. It can be shown that the dissipation of energy takes place on small scales, namely, scales smaller than the so-called Taylor length θ. On the other hand, the turbulence is generated by driving forces injecting energy into the flow on large scales, $l \geq L_0$, where L_0 is given by the correlation length of $u(x)$. Thus for the transition from $q(l, x)$ to $q(l', x)$ with $l' < l$, the same amount of energy is transferred to a scale l, as it is transferred from this scale to even smaller scales l'. The conservation of the transferred energy (more precisely the energy rate per unit volume) can be assumed as long as $L_0 > l, l' > \theta$. This range is called the inertial range, where the turbulent field develops independently from boundary conditions and dissipation effects. It has been proposed that in this range universal features of turbulence arise.

Kolmogorov proposed that the disorder of turbulence expressed by the statistics of $q(l, x)$ and its nth order moments $< q(l, x)^n >$ should depend only on transferred energy ϵ and the scale l: $< q(l, x)^n >= f(\epsilon, l)$. By simple dimensional arguments it follows that

$$< q(l, x)^n > \propto < \epsilon^{n/3} > l^{n/3}. \tag{3}$$

The simplest ansatz is to take ϵ as a constant, thus the Kolmogorov scaling $n/3$ is obtained [4]. Assuming a lognormal distribution for ϵ, i.e. not ϵ but $ln\ \epsilon$ has a Gaussian distribution, Kolmogorov and Oboukhov proposed for $< \epsilon^{n/3} >$ an additional scaling term, leading to the so-called intermittency [5] correction [6]

$$< q(l, x)^n > \propto\ l^{\xi_n} \quad \text{with} \quad \xi_n = \frac{n}{3} - \mu \frac{n(n-3)}{18} \quad \text{and} \quad n > 2 \tag{4}$$

with $0.25 < \mu < 0.5$ (for further details see [2]). The form of ξ_n has been heavily debated during the last decades.

Here we want to point out a general consequence of nonlinear scaling exponents ξ_n, namely, that the probability densities of $p(q(l, x))$ cannot be Gaussian but must change their forms with l. This point is easily seen, if the definition of the moments Eq. (2) is considered. From the scaling relation Eq. (4), we take the scale dependence of the variance of the distributions: $\sigma_l^2 :=< q(l, x)^2 > \propto\ l^{2\alpha}$ (for a general consideration, α is any real number, for turbulence it is close to 1/3.). Next, we express the integral of the nth moment (2) by the normalized variable $\tilde{q}(l, x) = q/\sigma_l$

$$< q(l, x)^n >= \sigma_l^n \int_{-\infty}^{\infty} (\frac{q}{\sigma_l})^n p(q) dq = \sigma_l^n \int_{-\infty}^{\infty} (\tilde{q})^n \tilde{p}(\tilde{q}) d\tilde{q}. \tag{5}$$

If the normalized probabilities $\tilde{p}(\tilde{q})$ do not change their form with l, like it would be the case if all probabilities are Gaussian, the integral on the right side is a constant and thus $< q(l, x)^n > \propto \sigma_l^n \propto l^{n\alpha}$ or the scaling indices ξ_n are linear in n. Saying this in another way, the nonlinear n- dependence of ξ_n means nothing else than that the probabilities $p(q(l, x))$ must change their form with the scale parameter l. In this way the discussion on the nonlinearity of ξ_n is linked to the discussion of non-Gaussian statistics. We see in Fig. 1 that for large scales ($l \approx L_0$) the distributions are nearly Gaussian. As the scale decreases, the probability densities become more and more fat tailed.

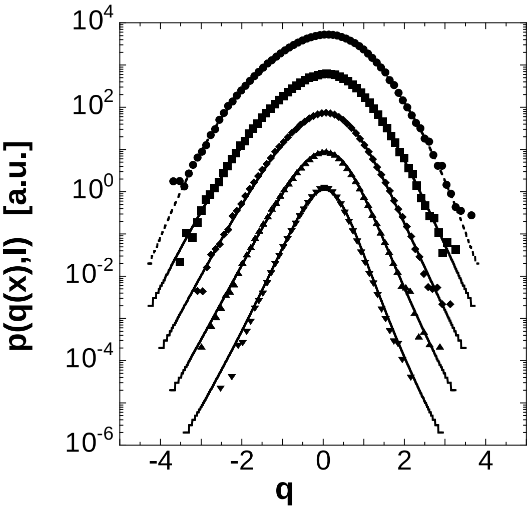

Fig. 1. Probability distributions obtained from a turbulent velocity signal measured in a free jet (bold symbols). The probabilities are compared with the numerical solution of the Fokker-Planck equation (solid lines). The scales l are (from top to bottom): $l = L_0$, $0.6L_0$, $0.35L_0$, $0.2L_0$ and $0.1L_0$. The distribution at the largest scale L_0 was parameterized (dashed line) and used as initial condition for the Fokker-Planck equation (L_0 is the correlation length of the turbulent velocity signal). The pdfs have been shifted in vertical direction for clarity of presentation and all pdfs have been normalized to their standard deviations; after [7].

Based on this finding, we can argue why a Lévy process, discussed above, is not consistent with the statistics of velocity increments of turbulence. As already mentioned, the Lévy statistics is based on the stability. Thus, the sum of the variables should have the same statistics. The sum of two successive increments $q(l,x) + q(l,x+l) = (u(x+l) - u(x)) + (u(x+2l) - u(x+l))$ is nothing else than the increment $q(2l,x)$ of a scale twice as big. We have just seen that $p(q(2l,x))$ has a different form, i.e. the process is not stable.

3 Finance

Next, we present some anomalous statistical features of data from the financial market. These features are astonishingly quite similar to the just discussed intermittency of turbulence [8] and are often called clustering of volatility (cf. [9–11]). The following analysis is based on a data set $Y(t)$, which consists of 1 472 241 quotes for US dollar-German Mark exchange rates from the years 1992 and 1993. Many of the features we will discuss here are also found in other financial data like for quotes of stocks. One central issue is the understanding of the statistics of price changes over a certain time interval τ which determines losses and gains. The changes of a time series of quotations $Y(t)$ are commonly measured by returns $r(\tau,t) := Y(t+\tau)/Y(t)$, logarithmic returns or by increments $q(\tau,t) := Y(t+\tau) - Y(t)$. The moments of these quantities display often power law behavior similar to the just discussed Kolmogorov scaling for turbulence, cf. [8,13–16]. In addition one finds for the probability distributions an increasing tendency to fat tail probability distributions for small τ (see Fig. 2) and [16]. This represents the high frequency dynamics of the financial market. The identification of the underlying process leading to these fat tail probability density functions of price changes is a prominent puzzle (see [9,14,15,17–19]), like it is for turbulence.

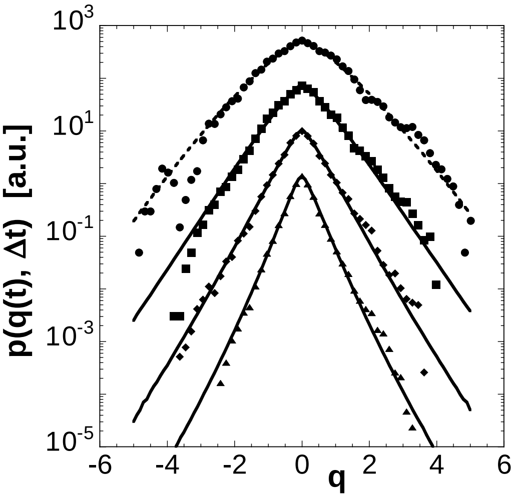

Fig. 2. Probability densities (*pdf*) $p(q(t),\tau)$ of the price changes $q(\tau,t) = Y(t+\tau) - Y(t)$ for the time delays $\tau = 5120, 10240, 20480, 40960s$ (from bottom to top). Symbols: results obtained from the analysis of middle prices of bit-ask quotes for the US dollar-German Mark exchange rates from October 1st, 1992 until September 30th, 1993. Full lines: results from a numerical iteration of the Fokker-Planck equation (14); the probability distribution for $\tau = 40960s$ (dashed lines) was taken as the initial condition. The pdfs have been shifted in vertical direction for clarity of presentation and all pdfs have been normalized to their standard deviations; after [12].

4 Surface roughness

As a last example of structures whose complexity has attracted a considerable amount of interest we discuss the surface roughness. Among the techniques used to characterize scale dependent surface roughness, the most prominent ones are the concepts of self-affinity and multi-affinity, where the multi-affine $f(\alpha)$ spectrum has been regarded as the most complete characterization of a surface [20–23]. Different definitions have been proposed to measure the scale dependent surface roughness. Here we are interested in the method of the analysis, thus we will process with a simple height increment,

$$q(l,x) := h(x + r/2) - h(x - r/2). \tag{6}$$

The meaning of left, mid or right centered increments will be discussed in [24]. In Fig. 3 the scale dependent distributions of the height increments are shown, which display nongaussian tails in a quite different structure of the fat tails (compare Figs. 1 2). The investigation based on multiscaling properties with respect to the moments of the height increments, suggest similarities to turbulence and financial data [24].

5 Stochastic Process for Scale Dependent Complexity

Based on the examples of complex structures and their analysis by means of the statistics of a scale dependent quantity $q(l,x)$, we want to report on an alternative, non conventional way to characterize anomalous statistics [7,12,25–28]. This method is based on the idea of a cascade process as discussed in the context of turbulence. In particular we show how from given data a stochastic process can be estimated. This makes it possible to model the measured statistics quite detailed. Guided by the finding that the statistics changes with scale, as shown in Figs. 1, 2 and 3, we consider the evolution of the quantity $q(l,x)$, or $q(\tau,t)$ with the variable l or τ respectively. Thus, we will consider processes which evolve in the scale variable. This we take as a stochastic cascade process.

For the following, we present a generalized discussion. Let us denote the given complex structure by $Y(x)$. With respect to the preceding sections, $Y(x)$ denotes the velocity of a turbulent flow at the location x, the price of a stock at the time "x", or the height at x. To characterize the disorder of the structure $Y(x)$ a local scale resolving quantity $q(l,x)$ is considered. This quantity q is here defined as an increment $q(l,x) := Y(x+l) - Y(x)$, but may also be a wavelet, a local measure, or the sum of the square of derivatives of Y, just to mention some possibilities. For time series, like the financial market data, l stands for the time lag. Note, the moments of $p(q)$ are called structure functions if q is an increment. It is easily seen that the second moment corresponds to the autocorrelation function $< Y(x)Y(x+l) >$, which

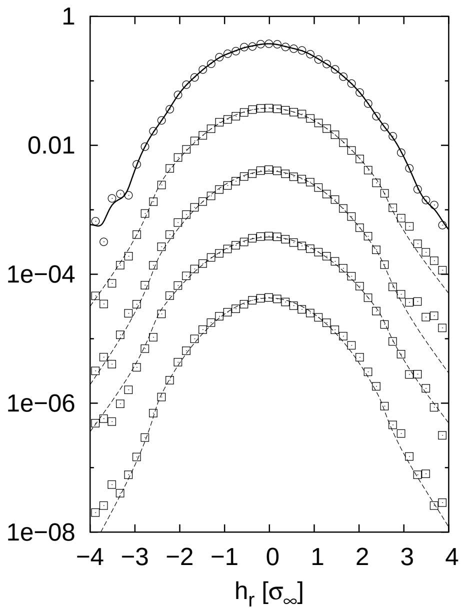

Fig. 3. Probability densities obtained from the surface roughness of a cobblestone road. Numerical solution of the integrated form of Fokker-Planck equation compared to empirical pdf (symbols) at different scales r. Solid line: empirical pdf parameterized at $r = 188mm$, dashed lines: reconstructed pdf. Scales r are $188, 158, 112, 79, 46mm$ from top to bottom. Pdf are shifted in vertical direction for clarity of presentation, after [27].

is related by the Wiener - Khintchine theorem to the power spectrum of $Y(x)$. Higher order moments correspond to higher order two point correlations.

For one fixed scale l we get the scale dependent disorder by the statistics of $q(l,x)$. More complete stochastic information of the disorder on all length scales is given by the joint probability density function

$$p(q_1, \ldots, q_n), \tag{7}$$

where we set $q_i = q(l_i, x)$. Without loss of generality we take $l_i < l_{i+1}$. This joint probability may be seen in analogy to joint probabilities of statistical mechanics (thermodynamics), describing in the most general way the occupation probabilities of the microstates of n particles, where q is a six-dimensional state vector.

Next, the question is discussed whether it is possible to simplify the joint probability. In general, such joint probabilities can be expressed by conditional probabilities:

$$p(q_1, \ldots, q_n) = p(q_1 \mid q_2, \ldots, q_n) p(q_2 \mid q_3, \ldots, q_n) \ldots p(q_{n-1} \mid q_n) p(q_n). \tag{8}$$

A simplification is given, if for the multiple conditioned probabilities

$$p(q_i \mid q_{i+1}, \dots, q_n) = p(q_i \mid q_{i+1}) \tag{9}$$

holds. Eq. (9) is known as the condition for a Markov process evolving from state q_{i+1} to the state q_i, i.e. from scale l_{i+1} to l_i.

A "single particle approximation", as used for the statistical mechanics of the ideal gas, would correspond to the condition:

$$p(q_1, \dots, q_n) = p(q_1)p(q_2)\dots p(q_n). \tag{10}$$

Based on equations (8) and (9), Eq. (10) holds if $p(q_i \mid q_{i+1}) = p(q_i)$. Only for the last case, the knowledge of $p(q_i)$ is sufficient to characterize the complexity of the whole system. Otherwise, an analysis of only $p(q_i)$, for the scale l_i, is a non unique projection of the whole complexity of the system. Such a projection is done by the investigations of the moments as presented in Eqs. (3) and (4). This is also the case for the characterization of complex structures by means of fractality or multiaffinity (cf. for multiaffinity [22], for turbulence [2,3], for financial market [13]). A much more general characterization of the disorder is obtained if the transition probabilities from one scale to a smaller one are investigated and if equation (9) is fulfilled.

In the following, we focus on the more complete statistical characterization proposed above. The basic idea is to consider the evolution of $q(l, x)$ at one fixed point x as a stochastic process in scale l. (The extensive discussion of the analysis of financial and turbulent data can be found in [7,12]).

To prove that a Markov process is given, it has to be shown that (9) holds. For a given data set $Y(x)$ it is easy to evaluate the corresponding conditional probabilities by the calculation of $q(l, x)$ at different points x for several fixed scales l_i. Having shown that the multiconditioned probabilities are equal to the single conditioned probabilities, the evolution of the conditional probability density $p(q, l \mid q_0, l_0)$ starting from the initial scale l_0 follows

$$-\,l\,\frac{\partial}{\partial l}\,p(q,l \mid q_0,l_0) \;=\; \sum_{k=1}^{\infty}\left(-\frac{\partial}{\partial q}\right)^{k} D^{(k)}(q,l)\,p(q,l \mid q_0,l_0). \tag{11}$$

(The minus sign on the left side is introduced, because we consider processes running to smaller scales l, furthermore we multiply the stochastic equation by l, which leads to a new parameterization of the cascade by the variable $ln(1/l)$, a simplification for a process with scaling law behavior of its moments.) This equation is known as the Kramers-Moyal expansion [29]. The Kramers-Moyal coefficients $D^{(k)}(q, l)$ are defined as the limit $\Delta l \to 0$ of the conditional moments $M^{(k)}(q, l, \Delta l)$:

$$D^{(k)}(q,l) \;=\; \lim_{\Delta l \to 0} M^{(k)}(q,l,\Delta l)\;,$$

$$M^{(k)}(q,l,\Delta l) \;:=\; \frac{l}{k!\,\Delta l}\int_{-\infty}^{+\infty} (\tilde{q}-q)^{k}\; p\,(\tilde{q}, l-\Delta l \mid q,l)\; d\tilde{q}. \tag{12}$$

Thus, for the estimation of the $D^{(k)}$ coefficients it is only necessary to estimate the conditional probabilities $p\,(\tilde{q}, l - \Delta l \mid q, l)$. For a general stochastic process, all Kramers-Moyal coefficients are different from zero. According to Pawula's theorem, however, the Kramers-Moyal expansion stops after the second term, provided that the fourth order coefficient $D^{(4)}(q,l)$ vanishes. In that case the Kramers-Moyal expansion is reduced to a Fokker-Planck equation (also known as Kolmogorov equation [30]):

$$- l \, \frac{\partial}{\partial l} \, p(q, l \mid q_0, l_0) \; = \; \left\{ - \frac{\partial}{\partial q} \, D^{(1)}(q,l) \; + \; \frac{\partial^2}{\partial q^2} \, D^{(2)}(q,l) \right\} \; p(q, l \mid q_0, l_0). \quad (13)$$

$D^{(1)}$ is denoted as drift term, $D^{(2)}$ as diffusion term. The probability density function $p(q,l)$ obeys the same equation:

$$- l \, \frac{\partial}{\partial l} \, p(q, l) \; = \; \left\{ - \frac{\partial}{\partial q} \, D^{(1)}(q,l) \; + \; \frac{\partial^2}{\partial q^2} \, D^{(2)}(q,l) \right\} \; p(q, l). \quad (14)$$

We remind the reader that the Fokker-Planck equation describes the probability density function of a stochastic process generated by a corresponding Langevin equation (we use Itô's definition)

$$- \frac{\partial}{\partial l} q(l) = D^{(1)}(q,l)/l + \sqrt{D^{(2)}(q,l)/l} \; f(l) \quad (15)$$

where $f(l)$ is white noise (Gaussian distributed and delta correlated). This feature provides an approach to simulate numerically typical data by the knowledge of the process coefficients, as it was proposed for example in [27] and shown in [28].

6 Discussion and conclusion

Let us return to the discussion of scale dependent complexity in turbulence, finance and surface roughness. We have reported on a way to characterize this complexity by means of stochastic processes. A new point is that we consider processes running in the scale variable and not in time as usually. The presented features of the stochastic process have been well known for decades [30]. A strength of this analysis is that it does not depend on assumptions. By the estimation of the conditional probabilities, the Markovian property (9) as well as the Kramers-Moyal coefficients (12) can be evaluated [31,32]. Knowing the evolution equation (11), the n-increment statistics $p(q_1, \ldots, q_n)$ is known, too. Definitely, an information like the scaling behavior of the moments of $q(l,x)$ can also be extracted from the knowledge of the process equations. Multiplying (11) by q^n and successively integrating over q_0 and q, an equation for the moments is obtained:

$$- l \, \frac{\partial}{\partial l} \, < q^n > = \sum_{k=1}^{n} \left(- \frac{\partial}{\partial q} \right)^k \frac{n!}{(n-k)!} < D^{(k)}(q,l) q^{n-k} > . \quad (16)$$

Scaling, i.e. multi affinity, is obtained if $D^{(k)}(q,l) \propto q^k$, see [33].

Based on this procedure we were able to reconstruct directly from the given data the corresponding stochastic processes. Knowing these processes one can perform numerical solutions (see [7,12,27]). In Figs. 1, 2 and 3 the numerical solutions are shown by solid curves. We see that the heavy tailed structure of the probabilities is well described by this approach over a Fokker-Planck equation. This result may be taken as a verification of the above described method to reconstruct the process from pure data analysis.

The knowledge of the stochastic process equations allows a detailed analysis. For financial as well as for the turbulent data it was found that the diffusion term is quadratic in the scale resolved variable q. With respect to the corresponding Langevin equation (15), the multiplicative nature of the noise term becomes evident. It is this multiplicative noise which causes fat tails of the probability densities and eventually multifractal scaling. The scale dependency of drift and diffusion term corresponds to a non-stationary process in scale variables τ and l, respectively. From this point we conclude and confirm our previous finding that a Lévy statistics for one fixed scale, i.e. for the statistics of $q(x.l)$ for fixed l, is not the adequate class of statistics. The true structure of the complexity is unveiled by the two scale statistics.

Comparing the tips of the distributions at small scales in Figs. 1 and 2 a less sharp tip for turbulence is found. This finding is in accordance with a larger additive term in the diffusion term $D^{(2)}$ for the turbulence data. Knowing that $D^{(2)}$ has an additive and quadratic q dependence it is clear that for small q values, i.e. for the tips of the distribution, the additive term dominates. Taking this result in combination with the Langevin equation, we see that for small q values Gaussian noise is present, which leads to a Gaussian tip of the probability distribution, as found in Fig. 1. Thus we see that the detailed analysis of the Fokker-Planck process allows to distinguish the complexity of turbulence and finance more profoundly, as it is the case for the multiscaling analysis where only different scaling indices are found. Here it should be noted, that the additive term in $D^{(2)}$ violates the proposed proper scaling behavior for turbulence (4). In [34] it was even shown that the stochastic process will change considerable with the Reynolds number, putting into question concepts on universal turbulence.

As a last comment we want to point out, that the results for the surface roughness shows, that also other forms of probability densities can be grasped quite well with this method. Thus the analysis with scale dependent stochastic process does not depend on scaling behavior or some special form of the probabilities. The basic features like Markov condition (9) can even be verified with the experimental data.

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