Stochastic modelling of fat-tailed probabilities of foreign exchange-rates

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Abstract

In a recent work [14] it has been shown that the statistics of price-changes on foreign-exchange rates measured by increments can be characterized completely by a Fokker-Planck-equation. The explicit form of this Fokker-Planck-equation was deduced directly from empirical data. Here we show that this result does not hold only for one specific construction of price-changes by increments but also for returns and logarithmic-returns, which are commonly used to quantify fluctuations in financial time-series over different time horizons. For all these quantities (increment and both kinds of returns) an explicit Fokker-Planck-equation is presented and a verification of the quality of this description is shown by the reproduction of fat-tailed probability density functions for different time scales. We propose this method as a generalization of multifractal analysis.

1 Introduction

Financial markets are amongst the most complex systems human society has brought up. Due to the high number of interactions between different agents, market segments and so forth an almost arbitrary level of complexity is reached. So it is only natural that methods from complex-system-theory and statistical sciences are more and more used to describe the phenomena related to this kind of market [1, 2, 3, 4, 5, 6, 7].

In this article we present a comprehensive analysis of a certain market segment, namely the foreign exchange (FX) market. We show how to model multi-scale statistics of price-changes in the FX market. *Price-changes* are generally described by means of relative changes, the so-called returns

$$r(t,\tau) := \frac{s(t+\tau) - s(t)}{s(t)} \qquad (1)$$

or the logarithmic-returns

$$l(t,\tau) := \ln \frac{s(t+\tau)}{s(t)}, \qquad (2)$$

or by absolute changes, the so called increments

$$i(t,\tau) := s(t+\tau) - s(t). \qquad (3)$$

Here *s(t)* denotes the time-series of FX market quotes. Please note that we chose a physical time-scale for *t* in contrast to often used artificial scales like business-time or ϑ-time. Consequently, any change over a certain time-scale t can only be considered for the statistics if there are actually two events $s(t_1)$ and $s(t_2)$ with the difference t_2-t_1 being exactly equal to τ.
For the general discussion for all three price-change descriptions we use $x(t, \tau)$. The use of logarithmic and non-logarithmic quantities emerges quite naturally from two different kinds of stochastic processes, namely those with multiplicative (return types) and additive (incremental description) characteristics, respectively. One of the key results, of the work presented here, is the stochastic equivalence of all three descriptions. This is of special importance, as many models used in financial analysis are based on return-type measures as a natural choice. We can demonstrate here, that at least the probability density functions of changes of all three types can be described with the same model and parameters being qualitatively equal. So from our viewpoint, there is no need to prefer return type measures to incremental changes. Whether this a special feature of FX rate data and might be different in the stock market would be subject to further analysis. Very interesting results concerning the use of different measures of change in the stock market are found in [24].

Our procedure to analyse the complexity of the FX market data is based on the idea that there is an analogy between the financial market and fully developed turbulence, as it was proposed by Ghashghaie et al. [11]. This analogy leads to the idea that also the financial market is governed by cascade-like processes, connecting the price-changes for different τ. Recently it has been shown that it is possible to extract a Fokker-Planck-equation by pure data analysis, which grasps the underlying cascade-like process [8, 9, 14]. The use of continuous interpolation of discrete multiplicative cascades to model financial data is presented in [10].

The central feature of the model presented here is the ability to give an equation for the development of the probability density functions (*pdf*) of the price-changes with respect to the *scale* τ. The only assumption is the presence of Markov properties in the stochastic process. The verification of this assumption is part of our analysis. We show that the resulting differential equation, i.e. Fokker-Planck-equation, is able to describe the well-known cross-over from a nearly normal pdf to strongly leptokurtic or fat-tailed probability densities when going from large-scales to shorter ones [7]. The description of these fat-tails is of greatest importance in various fields (c.f. [12]) as they represent the much higher probability of extreme events compared to a Gaussian or pure random process. One prominent example is the necessity for such models in modern risk management.
Here, we would like to refer to the work of Silva and Yakovenko [22], who also model the pdf of different financial data sets by means of a Fokker-Planck-equation obtained from proper analysis of the Heston model [23]. In contrast to their work, the analysis presented here does not rely on any specific model for the stochastic process of the FX rates we look at and is in so far a somewhat more general concept.

The article is organized as follows: In Section 2 a summary of basic features of Markov processes, which we use in our analysis, is presented. Section 3 is devoted to the data analysis, namely, evidence of Markov properties is given as well as the estimation of the Kramers-Moyal-coefficients. The verification of the quality of the estimated Fokker-Planck-equations is given in Section 4. The article is finished by some interpretations and discussions. All results demonstrated here are obtained from a high-frequency data set consisting of approximately 1,4 x 10^6 quotes of the German Mark – US Dollar exchange rate from October 1992 until September 1993 [20].

2 Theory of Markov processes

The concept of Markov properties is a central term in the analysis we present. Consequently, in this Section we give a short overview of the mathematics used in the description of the pdfs of FX data. Further details may be found in [21] and [14]. First of all, the basic quantity of a Markov process, the conditional probability is defined as

$$p(x_1,\tau_1|x_2,\tau_2)=\frac{p(x_1,\tau_1;x_2,\tau_2)}{p(x_2,\tau_2)}, \tag{4}$$

where $p(x_1,\tau_1; x_2,\tau_2)$ denotes the joint probability to find the price-change x_1 over the time scale τ_1 and the price-change x_2 over the corresponding scale τ_2, simultaneously. (Note, for our investigation of the τ-evolution of one price-change we always regard this for one fixed time t. The statistics then is obtained for different times t.) The higher order conditional probabilities are given accordingly by

$$p(x_1,\tau_1|x_2,\tau_2;x_3,\tau_3;...;x_n,\tau_n)=\frac{p(x_1,\tau_1;x_2,\tau_2;x_3,\tau_3;...;x_n,\tau_n)}{p(x_2,\tau_2;x_3,\tau_3;...;x_n,\tau_n)}. \tag{5}$$

For convenience we declare that scales τ are increasing with higher indices so that $\tau_1 < \tau_2 < \tau_3$ … The condition for a process having Markov properties can be written as

$$p(x_1,\tau_1|x_2,\tau_2;x_3,\tau_3;...;x_n,\tau_n)=p(x_1,\tau_1|x_2,\tau_2) \tag{6}$$

for all $n \geq 3$. A most important consequence of (6) is that any n-point joint probability (in our case the term "n-scale-joint-pdf" may be more appropriate) can now be expressed by simple conditional probabilities as

$$p(x_1,\tau_1;x_2,\tau_2;...;x_n,\tau_n)=p(x_1,\tau_1|x_2,\tau_2)p(x_2,\tau_2|x_3,\tau_3)..p(x_{n-1},\tau_{n-1}|x_n,\tau_n)p(x_n,\tau_n) \tag{7}$$

Thus, for a Markov process, the knowledge of the simple conditional probabilities $p(x,\tau| x_0,\tau_0)$ for arbitrary scales τ, τ_0 with $\tau < \tau_0$ is equivalent to the knowledge of any arbitrary n-point joint probability.
Next, we discuss the presentation of Markov processes by differential equations. The scale evolution of conditional probability densities is described by the so called Kramers-Moyal-expansion [21]

$$-\tau\frac{\partial}{\partial\tau}p(x,\tau|x_0,\tau_0)=\sum_{k=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^k D_k(x,\tau)p(x,\tau|x_0,\tau_0), \tag{8}$$

where the coefficients D_k [17] are defined as limits $\Delta\tau \to 0$ of the conditional moments

$$M_k(x,\tau,\Delta\tau) = \frac{\tau}{k!\,\Delta\tau}\int_{-\infty}^{\infty}(\tilde{x}-x)^k\, p(\tilde{x},\tau-\Delta\tau\,|\,x,\tau)\,d\tilde{x}. \quad (9)$$

In general, all Kramers-Moyal-coefficients D_k are non-zero. If, however, the fourth coefficient $D_4(x,\tau)$ vanishes, Pawula's theorem [21] states that only the coefficients $D_1(x,\tau)$ and $D_2(x,\tau)$ are non-zero. $D_1(x,\tau)$ is called drift-term and $D_2(x,\tau)$ diffusion-term. All higher coefficients D_k with $k \geq 3$ must vanish. Thus, the Kramers-Moyal-expansion (8) reduces to a Fokker-Planck-equation of the form

$$-\tau\frac{\partial}{\partial\tau}p(x,\tau|x_0,\tau_0) = \left[-\frac{\partial}{\partial x}D_1(x,\tau) + \frac{\partial^2}{\partial x^2}D_2(x,\tau)\right]p(x,\tau|x_0,\tau_0) \quad (10)$$

Note that the same Fokker-Planck-equation is valid for the non-conditioned probabilities $p(x,\tau)$ as can be seen by integration over the condition. The pre-factor $-\tau$

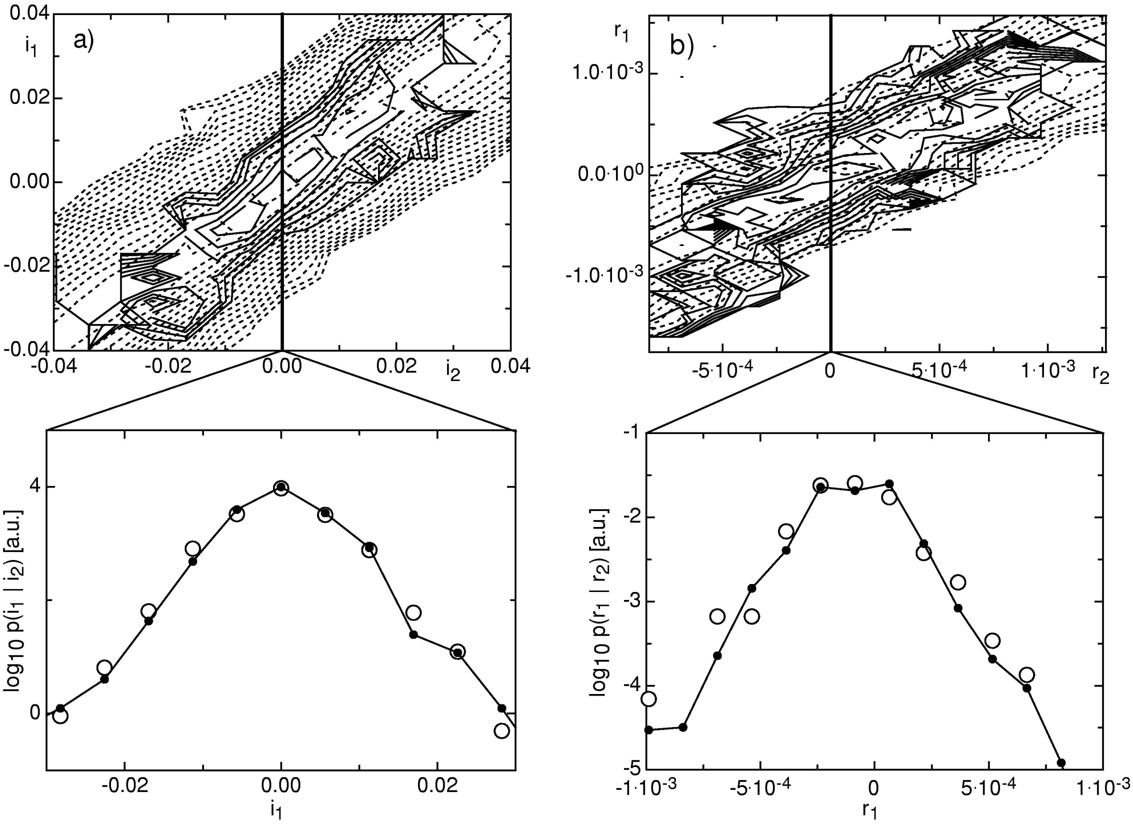

Figure 1: Contour plot of single and double conditioned probabilities (see eq. (11)) of incremental changes *i(t,τ)* (a) and returns *r(t,τ)* (b) for τ_1=1 hour, $\tau_2 = \tau_1 + \Delta\tau$ and $\tau_3 = \tau_1 + 2\Delta\tau$ with $\Delta\tau$= 4 min. The bottom plots are cuts through the contour plots at given lines of constant i_2 and r_2, respectively.

on the left side is due to the fact that we investigate a cascade from large towards smaller scales.

3 Data analysis

3.1 Data preparation

In [14] it was found that the complexity of the analysed data set can be divided up into two main parts. Fluctuations on time-scales $\tau \geq$ *4 min* are due to a stochastic cascade process whereas fluctuations on shorter time-scales are due to a different noise source which is independent of the cascade-process. This short-time noise acts like a measurement noise which is added later on to the data resulting from the stochastic cascade-process. As was demonstrated in [14] this measurement noise can be removed by applying a weighted moving average with a time constant of 44s over the whole time series. The relevant stochastic features of the cascade-process are not affected by this averaging procedure as far as our analysis is concerned.

3.2 Markov properties

In the preceding section it was stated that the only assumption necessary for the analysis presented here is the presence of Markov properties in the data. To prove this in a mathematically rigorous way, equation (6) needed to be verified for any positive value of *n* and for all possible combinations of scales τ. This is evidently not possible with a finite data set. As a strong hint towards Markov properties we evaluate special conditional probabilities for the case of *n=3*. That is, we check the equality

$$p(x_1,\tau_1|x_2,\tau_2;x_3,\tau_3) = p(x_1,\tau_1|x_2,\tau_2) \tag{11}$$

for different combinations of the τ_i and the x_i. For the sake of presentation (see Fig. 1), the time scales t_i are fixed as well as the third price-change x_3 which is held constant at $x_3=0$. The only remaining independent variables in (11) are x_1 and x_2. In this way a graphical representation of eq. (11) is possible by means of a contour plot. As can be seen from Fig. 1 both probabilities coincide quite well, for the increment changes $i(t,\tau)$ (a) as well as for the returns $r(t,\tau)$ (b). The same behaviour is observed for the already mentioned logarithmic-returns.

Of course this is only one singular realization of (11) but similar results are obtained for other sets of scales τ_i as far as $\Delta\tau$ remains larger than 4 minutes. So we take this as a good hint for the validity of Markov properties in the scale behaviour of conditional probabilities for all three types of price-changes discussed here.

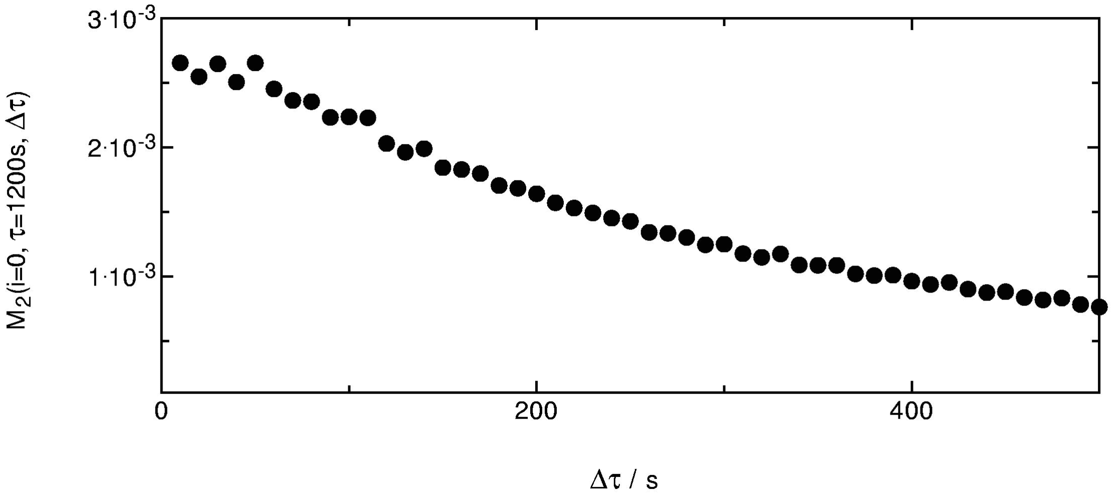

Figure 2: The conditional moment $M_2(x,\tau,\Delta\tau)$ for fixed values of x=0 and t=1200 s. This example is calculated from increments, the analogous behaviour is observed for return-type changes. It can be seen that the limit for $\Delta\tau \to 0$ can be realized by fitting the moments with a straight line towards the ordinate.

3.3 Coefficients of the Fokker-Planck-equation

The next step in estimating an analytical expression for the Fokker-Planck-equation (10) is to compute the coefficients $D_1(x,\tau)$ and $D_2(x,\tau)$. As can be seen from eqns. (9) and (4) this can be done by evaluating the joint probabilities $p(\tilde{x},\tau-\Delta\tau;x,\tau)$. Those are easily obtained from the data by counting the common occurrences of the

changes $\tilde{x}$ and x for the same times t. This enables us to estimate the conditional moments $M_k(x,\tau,\Delta\tau)$. To clarify how the limit $\Delta\tau \to 0$ is mastered to obtain the coefficients $D_k(x,\tau)$ we show in Fig. 2 an example of a conditional moment $M_2(x,\tau,\Delta\tau)$. This plot is representative for other scales and also for the shape of $M_1(x,\tau,\Delta\tau)$ and shows that the limit can be made by a linear regression over $\Delta\tau$. The constant part at $\Delta\tau = 0$ is then our best estimation for the Kramers-Moyal-coefficient $D_k(x,\tau)$. It has also been attempted to improve the process by fitting the moments with a power-law, but no essential improvement can be observed by doing so. For the meaning of higher-order corrections see the discussion in [18, 19].

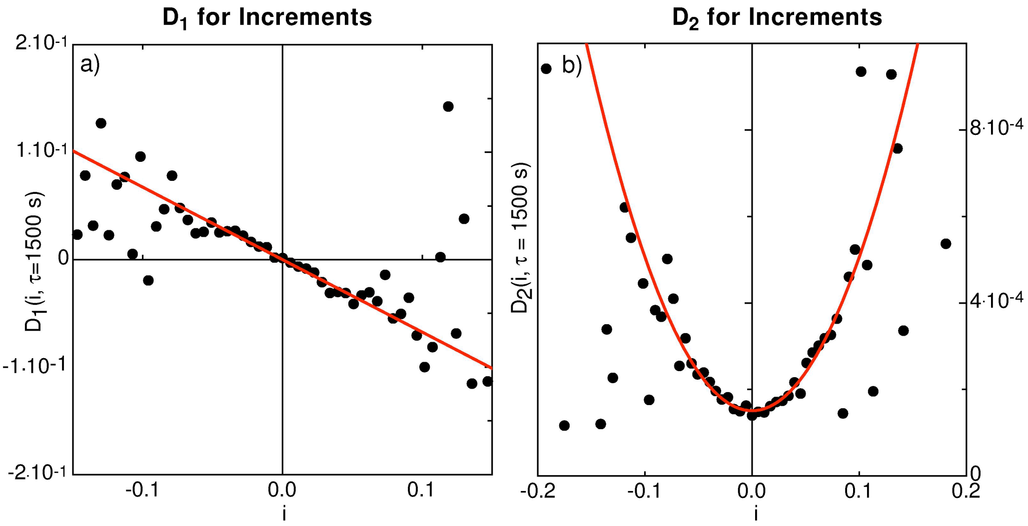

Figure 3: Functional relation of the Kramers-Moyal-coefficients $D_1(i,\tau=1500s)$ (a) and $D_2(i,\tau=1500s)$ (b), fluctuations are described in terms of increments $i(t,\tau)$. It can bee seen that the drift-term D_1 can be described by a straight line through the origin while the diffusion-term D_2 is best fitted with a polynomial of degree two without a linear part.

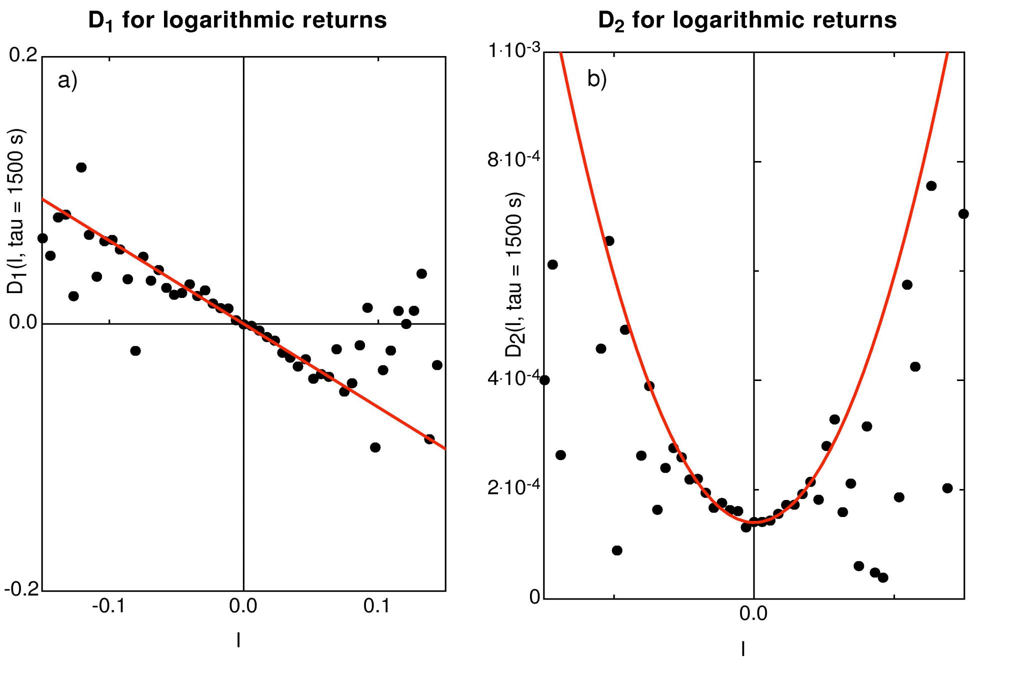

Figure 4: Kramers-Moyal-coefficients $D_1(l,\tau=1500s)$ (a) and $D_2(l,\tau=1500s)$ (b) for logarithmic returns. Otherwise like Fig. 3.

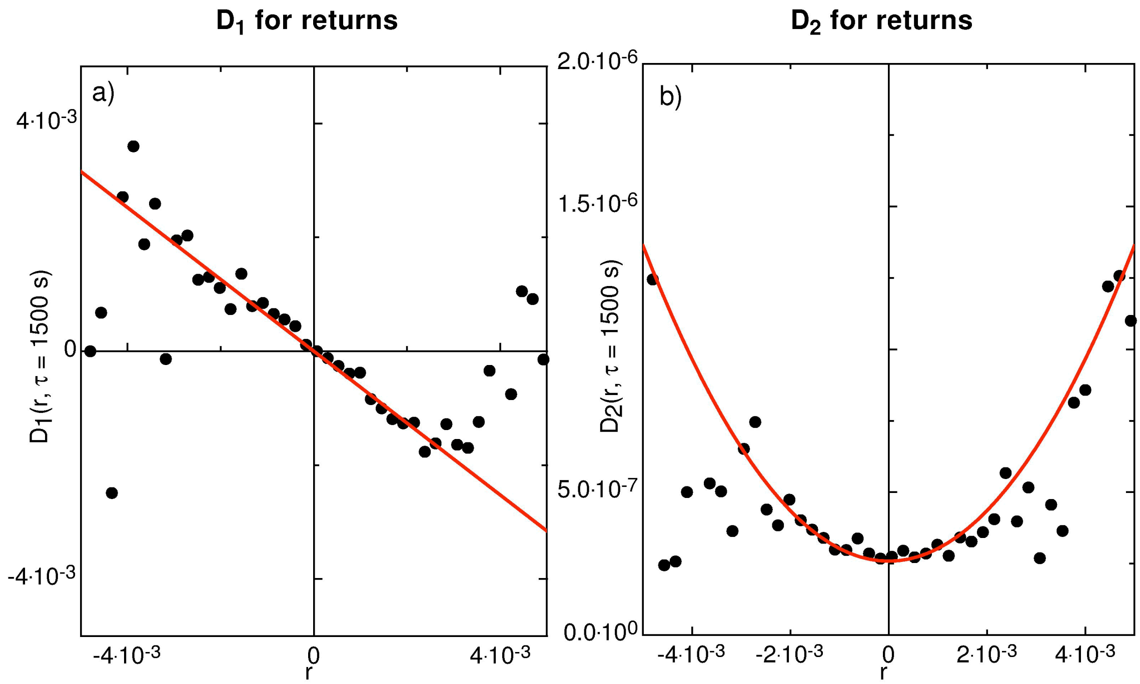

Figure 5: Kramers-Moyal-coefficients $D_1(r,\tau=1500s)$ (a) and $D_2(r,\tau=1500s)$ (b) for returns $r(t,\tau)$. Otherwise like Fig. 3.

In Figs. 3 – 5 the obtained coefficients for discrete time-scales τ are shown as functions of the price-changes *x*, i.e. for increments $I(t,\tau)$, log-returns $l(t,\tau)$ and returns $r(t,\tau)$. It can be clearly seen from the plots that for all three types of price-changes the Kramers-Moyal-coefficients can be described with a similar functional form. This is also true for different scales τ. Thus, these coefficients can be parametrized as

$$\begin{aligned} D_1(x,\tau) &= \alpha(\tau)x \\ D_2(x,\tau) &= \beta(\tau)+\gamma(\tau)x^2. \end{aligned} \tag{12}$$

The final step towards the Fokker-Planck-equation, Eq. (10), remains in finding the functional forms of the parameters $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$. In Fig. 6 the time-scale dependence of these parameters is shown for all three types of price-changes. From the figures we can see that both the slope, $\alpha(\tau)$, of $D_1(x,\tau)$ and the second-order term, $\gamma(\tau)$, of $D_2(x,\tau)$ are approximately constant while the constant part, $\beta(\tau)$, of $D_2(x,\tau)$ obviously exhibits a clear linear dependence of the time-scale τ. In a first approximation $\alpha(\tau)$ and $\gamma(\tau)$ can be taken to be the same for all three different types of price-changes, whereas the additive term $\beta(\tau)$ differs for returns $r(t,\tau)$.
Now, we have a complete description of the Fokker-Planck-equation (10) which describes, as we claim, the whole evolution of the probability density function of the changes of the observed FX rates. Also all moments are known. In the next section we demonstrate the effectiveness of this description.

4 Reproduction of probability densities

In this section we give evidence that the Fokker-Planck-equation, we deduced from the data set, contains all the information of the probability density functions $p(x,\tau)$

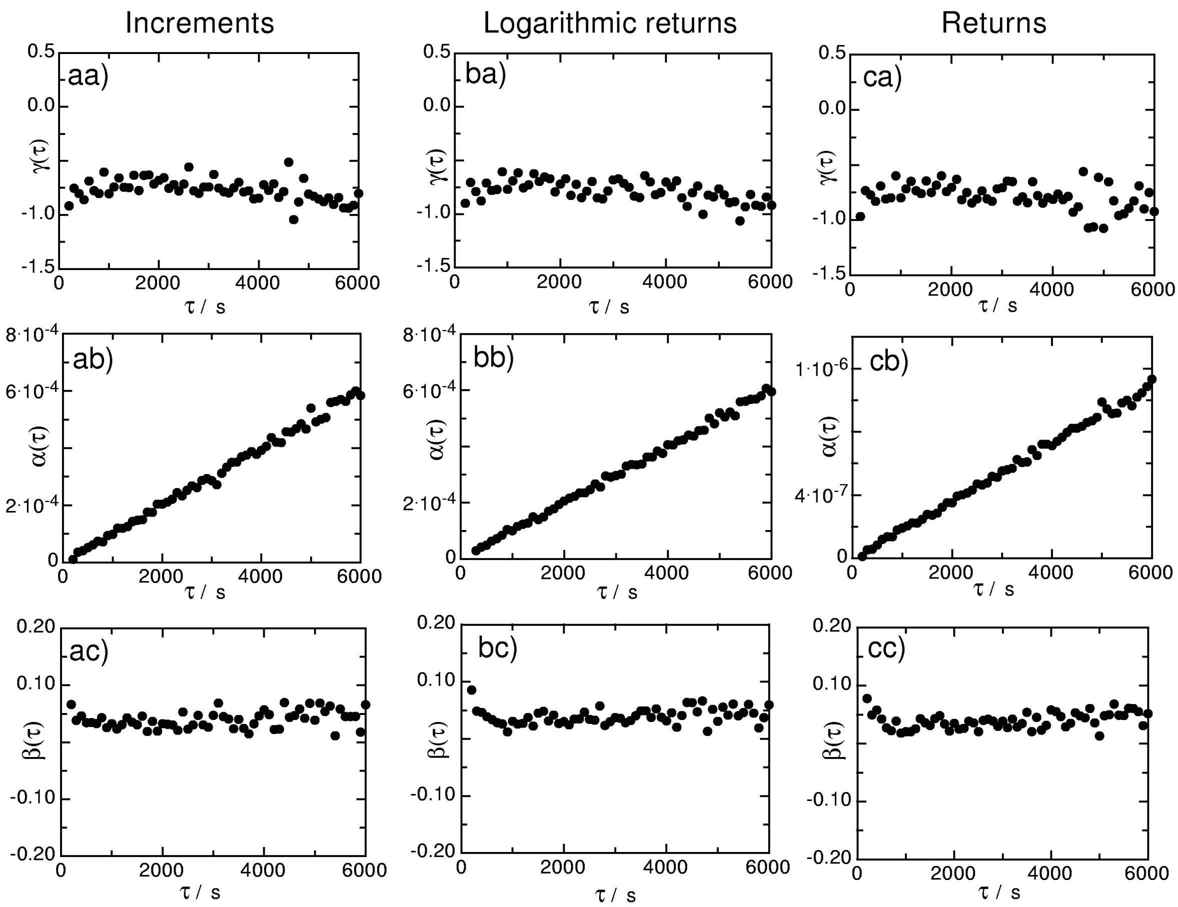

Figure 6: The dependence of the parameters γ, α and β on the time scale τ. The linear form of the constant term, α, of $D_2(x,\tau)$ (middle row, second index (b)) is clearly seen. Both other parameters can be approximated as constant values. The left column (first index (a)) shows the results for increments *$i(t,\tau)$*, the middle one (first index (b)) for log-returns *$l(t,\tau)$* and the one on the right (c) for returns *$r(t,\tau)$*. Second indices (a), (b) and (c) correspond to the three different parameters γ, α and β, resp. The stochastic equality of all three descriptions is evident.

over arbitrary time scales τ. Therefore, we solve the equation numerically and compare its solution with empirically gained pdfs on selected scales τ. For the numerical iteration of the Fokker-Planck-equation we need a starting distribution. Here we take the measured pdf from the time scale *τ= 43200s = 12 hours*. These values are fitted by a spline-function and discretized on the desired mesh. The pdfs computed from the Fokker-Planck-equation are shown in Fig. 7 in comparison with the empirically measured densities on the corresponding time-scales. The topmost pdf is the starting-distribution given to the algorithm. Clearly, the Fokker-Planck-equation is able to describe the evolution of the pdfs for all time-scales and for each of the three types of price-changes: increments (a), logarithmic-returns (b) and also normal returns (c). Also the cross-over to strongly fat-tailed distributions on shorter time-scales is reproduced. This result we take as a strong hint that the cascade Markov process is sufficiently described by a Fokker-Planck-equation and no higher-order Kramers-Moyal-coefficients (see Eq. 8) need to be considered.
As discussed in section 2 the same Fokker-Planck-equation is valid for any arbitrary conditional distribution. Thus, our results also contain the information of all conditional probability functions and thus of any n-scale joint-pdf.

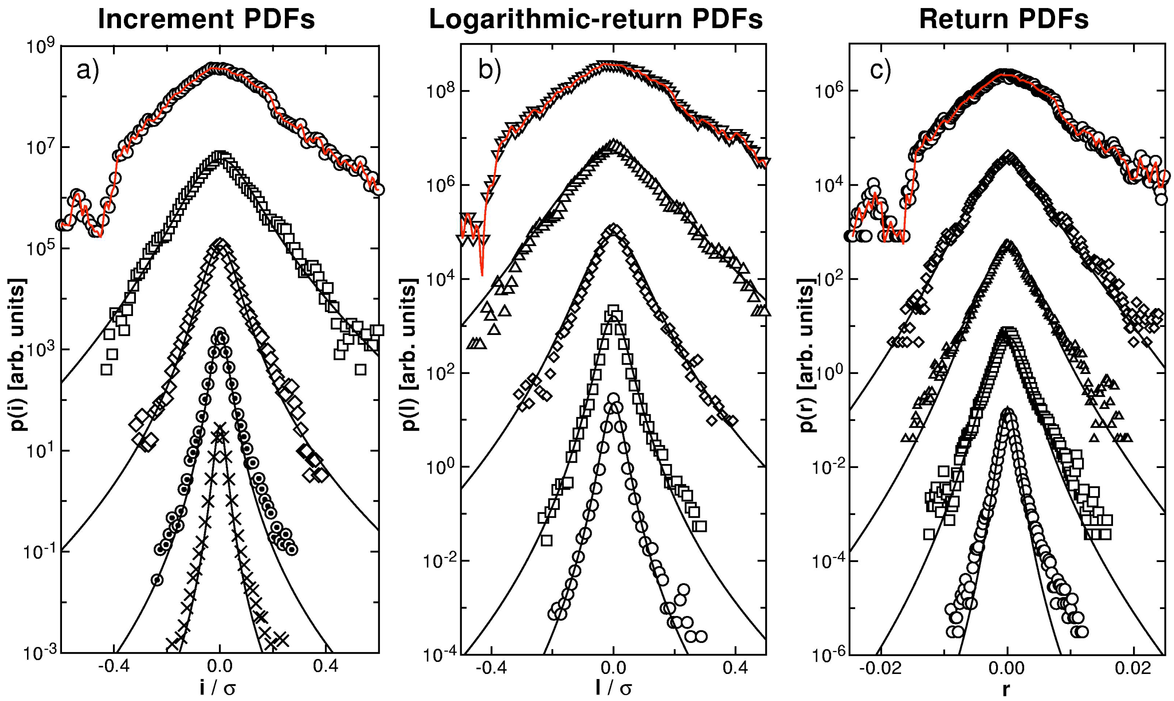

Figure 7: Comparison of the numerical solutions of the obtained Fokker-Planck-equations (solid lines) with the empirically found distributions (represented by symbols) of incremental changes, logarithmic-returns and returns (a), (b) and (c), resp. Note that the topmost curve is a spline-fit through the empirical data on the corresponding time-scale and is needed as starting condition for the solution of the Fokker-Planck-equation. Obviously, the Fokker-Planck-equation is capable of reproducing the evolution over the time-scale τ of the probability density functions of all three types of price-changes. Scales vary from 12 hours (top) to six minutes (bottom). The pdfs are shifted vertically by arbitrary factors for clarity of presentation.

5 Conclusions and Outlook

When viewed as a cascade-like process, the evolution of probability density functions of price-changes in the exchange rate of two selected currencies obey a Markov process. This enables us to set up a Fokker-Planck-equation to describe that evolution continuously over the time-scale τ. We demonstrated that it is possible for different commonly used measures of FX price-changes to obtain the coefficients of that equation by data analysis, namely the determination of conditional moments. Further it has been demonstrated that the resulting equation is truly capable of reproducing the measured distributions very precisely.
The method of characterization presented here offers a deeper insight into the nature of complexity because it comprises any n-scale joint-pdf. We want to point out that our results also contain the information from frequently used multiscaling analysis. To show this, one has to deduce from the Fokker-Planck-equation the evolution equation for the moments. As can be seen easily (multiplying Eq. (10) by x^n and successive partial integration of the Fokker-Planck-equation) this leads to

$$-\tau\frac{\partial}{\partial\tau} < x^n(\tau) >= n < D_1(x,\tau)x^{n-1}(\tau) > +n(n-1) < D_2(x,\tau)x^{n-2}(\tau) >, \qquad (13)$$

which results for equation (12) in

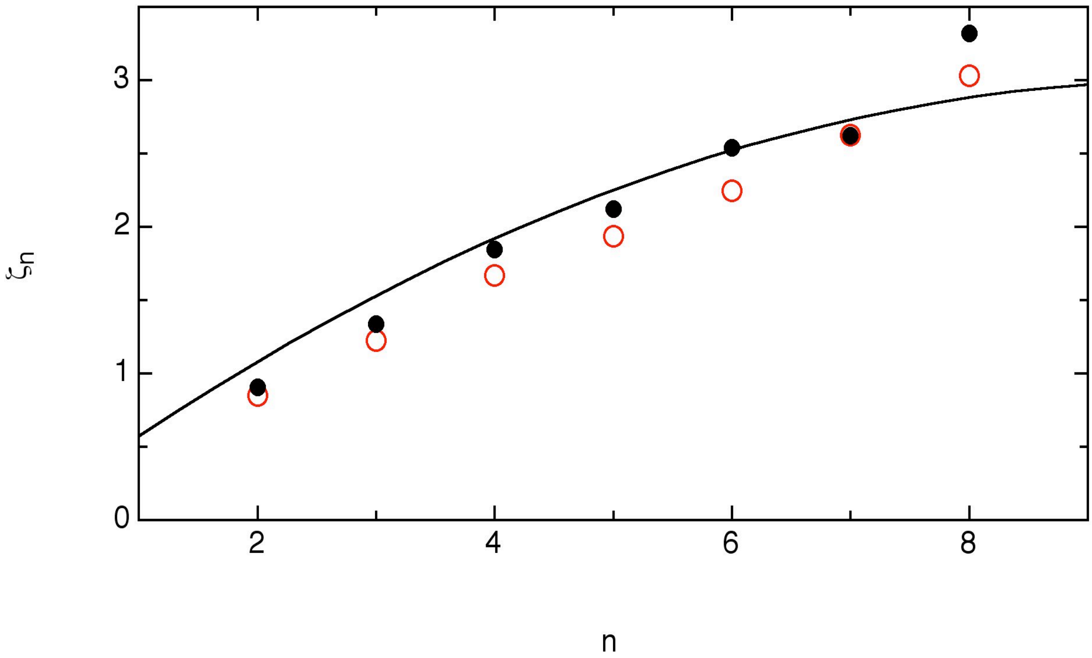

Figure 8: Scaling exponents ξ_n calculated from the data (open red circles) and from the reproduced PDF-solutions of the Fokker-Planck-equation (solid black circles). The curve is our prediction according to eq. 15.

$$-\tau\frac{\partial}{\partial\tau} < x^n(\tau) >= (-n\gamma(\tau) + n(n-1)\delta(\tau)) < x^n(\tau) > +n(n-1)\alpha(\tau) < x^{n-2}(\tau) > . \quad (14)$$

Neglecting the term containing $\alpha(\tau)$ and taking $\gamma(\tau)$ and $\delta(\tau)$ as constant this equation corresponds to a multiscaling behaviour of the moments with scaling exponent ξ_n:

$$< x^n > \propto \tau^{\xi_n}, \quad \xi_n = n\gamma - n(n-1)\delta \quad (15)$$

The functional form of ξ_n with γ and δ from the analysis presented here is shown in Fig. 8 compared to the actual values of the ξ_n deduced from the solutions of the Fokker-Planck equation (10) as well as from the empirical values. It is shown that in the statistically significant range up to about the sixth moment, the form of the ξ_n could easily be taken as linear, the quadratic form that follows from (15) expresses itself more clearly only in the higher moments with γ being about an order of magnitude larger than δ. Those moments of higher order have in our opinion a very limited signifinace due to the finiteness of the dataset.
In [14] and [13] it is stated that the parameters of the Fokker-Planck-equation and the scaling exponent are related to $\mu = \gamma/2\delta$, where μ characterizes a power-law of the tails of the pdfs. The results obtained here are in accordance with [14]. Note that the smallness and the errors of δ may vary μ between 4 and 8. This discrepancy to accepted values of 3 to 5 may be due to the instationarity of the coefficients. A further discussion of multifractality in Fokker-Planck dynamics is given in [15]. Note that, actually, the parameters of the Fokker-Planck-equation, especially $\gamma(\tau)$, are partially dependent on τ so that we have a non-stationary form of that equation. This may be taken as a hint towards the necessity of non-equlibrium descriptions as proposed in [16].

For the multiscaling analysis only the knowledge of the unconditional pdf $p(x,\tau)$ is necessary. The description of conditional probabilities has a higher information content than that of unconditioned ones, the difference of which we are about to show in current works. Thus, we conclude that the stochastic analysis presented here is more powerful than classical multiscaling or multifractal procedures. It is not only restricted to the analysis of the complexity of foreign exchange data but may be useful for any complex system involving multiscale properties.

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